Can one see the number of colors in $\eta, \eta' \to \pi^+\pi^-\gamma$?

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Abstract

We investigate the decays $\eta, \eta' \to \pi^+\pi^-\gamma$ up to next-to-leading order in the framework of the combined $1/N_c$ and chiral expansions. Counter terms of unnatural parity at next-to-leading order with unknown couplings are important to acommodate the results both to the experimental decay width and the photon spectrum. The presence of these coefficients does not allow for a determination of the number of colors from these decays.

PACS: 12.39.Fe

Keywords: Chiral Lagrangians, large N_c, chiral anomalies.

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The anomalous decay $\pi^0 \to \gamma\gamma$ is presented as a textbook example to confirm from low-energy hadron dynamics the number of colors to be $N_c = 3$, see e.g. [1], since this decay originates at tree level from the Wess-Zumino-Witten (WZW) term [2, 3] with a quantized prefactor N_c. The decay width $\Gamma_{\pi^0\to\gamma\gamma}$ is thus proportional to N_c^2, being quite sensitive to the number of colors, and in fact the result for $N_c = 3$ is in perfect agreement with experiment.

Recently, however, it was shown in [4, 5] that the cancellation of triangle anomalies in the standard model with an arbitrary number of colors leads to N_c dependent values of the quark charges, such that the vertex with one pion and two photons is completely canceled by the N_c dependent part of a Goldstone-Wilczek term [5, 6]. Within this scenario the decay $\pi^0 \to \gamma\gamma$ cannot be utilized to support $N_c = 3$. A similar cancellation also occurs for the decay $\eta \to \gamma\gamma$, if one neglects η-η' mixing. The N_c independence is maintained at one-loop order, i.e. at next-to-next-to-leading order in the combined chiral/large N_c expansion, for both the π^0 and the η decay, but the strong N_c dependence of the singlet decay $\eta_0 \to \gamma\gamma$ induces also a strong N_c dependence for $\eta \to \gamma\gamma$ due to η-η' mixing [7]. One concludes then that both the η and the η' decay show clear evidence that we live in a world with three colors.

On the other hand, it has been pointed out in [5] that at tree level the decay width of the process $\eta \to \pi^+\pi^-\gamma$ is proportional to N_c^2 and should replace the textbook process $\pi^0 \to \gamma\gamma$ lending support to $N_c = 3$. In analogy to the two-photon decays, the effects of η-η' mixing along with the inclusion of subleading contributions must be treated systematically, in order to make a rigorous statement on a possible determination of the number of colors from this process. In the present work we will therefore investigate the decays $\eta, \eta' \to \pi^+\pi^-\gamma$ up to next-to-leading order within the framework of large N_c chiral perturbation theory (ChPT) [8].

At leading order in the combined chiral and $1/N_c$ expansions the decays $\eta, \eta' \to \pi^+\pi^-\gamma$ originate from a piece in the WZW Lagrangian

$$S_{WZW}(U, v) = -\frac{N_c}{48\pi^2}\int \langle \Sigma_L^3 v - \Sigma_R^3 v\rangle, \tag{1}$$

where $\Sigma_L = U^\dagger dU$, $\Sigma_R = U dU^\dagger$, and we adopted the differential form notation of [8],

$$v = dx^\mu v_\mu, \qquad d = dx^\mu \partial_\mu \tag{2}$$

with the Grassmann variables dx^μ which yield the volume element $dx^\mu dx^\nu dx^\alpha dx^\beta = \epsilon^{\mu\nu\alpha\beta} d^4x$. The brackets $\langle \ldots \rangle$ denote the trace in flavor space, while the unitary matrix $U = e^{i\phi}$ collects the pseudoscalar meson nonet (π, K, η_8, η_0). The external vector field $v = -eQA$ contains the photon field $A = A_\mu dx^\mu$ and the quark charge matrix Q of the u- d- and s-quarks which is usually assumed to be independent of the number of colors with $Q = \frac{1}{3}\mathrm{diag}(2, -1, -1)$. However, the cancellation of triangle anomalies requires Q to depend on N_c [4, 5]

$$\begin{aligned} Q &= \frac{1}{2}\mathrm{diag}\Big(\frac{1}{N_c} + 1, \frac{1}{N_c} - 1, \frac{1}{N_c} - 1\Big) \\ &= \hat{Q} + \Big(1 - \frac{N_c}{3}\Big)\frac{1}{2N_c}\mathbb{1} \end{aligned} \tag{3}$$

with $\hat{Q} = \frac{1}{3}\mathrm{diag}(2, -1, -1)$ being the conventional charge matrix, while the second term is proportional to the baryon number and gives rise to the Goldstone-Wilczek term. The anomalous Lagrangian of Eq. (1) decomposes into the conventional WZW Lagrangian of the $U(3)$ theory

with the charge matrix $\hat{Q}$ and a Goldstone-Wilczek term which vanishes for $N_c = 3$

$$S_{WZW}(U, v) = S_{WZW}(U, \hat{v}) + \Big(1 - \frac{N_c}{3}\Big) S_{GW}(U, A) \tag{4}$$

with $\hat{v} = -e\hat{Q}A$ and

$$\begin{aligned} S_{WZW}(U, \hat{v}) &= \frac{N_c e}{48\pi^2} \int \big\langle (\Sigma_L^3 - \Sigma_R^3)\, \hat{Q} \big\rangle A, \\ S_{GW}(U, A) &= \frac{e}{48\pi^2} \int \big\langle \Sigma_L^3 \big\rangle A. \end{aligned} \tag{5}$$

However, this presentation is not convenient to perform calculations within the framework of large N_c ChPT. To this end, one rather expands the quark charge matrix Q in powers of $1/N_c$

$$Q = \frac{1}{2}\mathrm{diag}(1, -1, -1) + \frac{1}{2N_c}\mathbb{1} \equiv Q^{(0)} + Q^{(1)}, \tag{6}$$

where the superscript denotes the order in the combined large N_c and chiral counting scheme, i.e. $Q^{(0)}$ ($Q^{(1)}$) is of order $\mathcal{O}(1)$ ($\mathcal{O}(\delta)$). From S_{WZW} one obtains the tree level contributions

$$\begin{aligned} S_{WZW}(U, v) = \int d^4x\, \mathcal{L}_{WZW} &= -\frac{iN_c e}{24\pi^2} \int \langle d\phi\, d\phi\, d\phi\, Q \rangle A \\ &= -\frac{iN_c e}{24\pi^2} \int \langle d\phi\, d\phi\, d\phi\, Q^{(0)} \rangle A, \end{aligned} \tag{7}$$

since for the processes $\eta_8, \eta_0 \to \pi^+\pi^-\gamma$ the trace with $Q^{(1)}$ in Eq. (7) vanishes. The pertinent amplitudes have the structure

$$\mathcal{A}^{(tree)}(\phi \to \pi^+\pi^-\gamma) = -\frac{N_c e}{\sqrt{3}\, 12\pi^2 f^3} k_\mu \epsilon_\nu p^+_\alpha p^-_\beta \epsilon^{\mu\nu\alpha\beta} \alpha_\phi^{(tree)}, \tag{8}$$

where $p^+(p^-)$ is the momentum of the outgoing $\pi^+(\pi^-)$ and $k(\epsilon)$ is the momentum (polarization) of the outgoing photon. Next we replace f^3 by $F_\phi F_\pi^2$ in Eq. (8) with the decay constants F_φ defined via

$$\langle 0 | \bar{q}\gamma_\mu\gamma_5 \lambda^i q | \varphi \rangle = i\sqrt{2}\, p_\mu F^i_\varphi \tag{9}$$

which is consistent at leading order. Neglecting η-η' mixing for the moment, the coefficients $\alpha_\phi^{(tree)}$ read

$$\alpha_\eta^{(tree)} = 1\,, \qquad \alpha_{\eta'}^{(tree)} = \sqrt{2}\,. \tag{10}$$

These are the expressions which were suggested to be utilized for a determination of N_c [5] [3]. Employing the experimental values [9]

$$\begin{aligned} \Gamma_{\eta \to \pi^+\pi^-\gamma} &= 56.1 \pm 5.4\ \mathrm{eV}, \\ \Gamma_{\eta' \to \pi^+\pi^-\gamma} &= 59.6 \pm 5.2\ \mathrm{keV}, \end{aligned} \tag{11}$$

we extract from the η decay $N_c = 7$ and $N_c = 10$ from the η' decay which is clearly in contradiction to the well-established value $N_c = 3$.

[3] Note, however, that a factor of 1/3 is missing in the amplitudes given in [5].

Taking into account η-η' mixing at leading order

$$\begin{aligned}\eta_8 &= \cos\vartheta^{(0)}\ \eta + \sin\vartheta^{(0)}\ \eta' \\ \eta_0 &= -\sin\vartheta^{(0)}\ \eta + \cos\vartheta^{(0)}\ \eta'\end{aligned} \tag{12}$$

with the mixing angle $\vartheta^{(0)}$ given by

$$\sin 2\vartheta^{(0)} = -\frac{4\sqrt{2}}{3}\frac{m_K^2 - m_\pi^2}{m_{\eta'}^2 - m_\eta^2} \tag{13}$$

the experimental values given in Eq. (11) allow either for $N_c = 4$ or $N_c = 5$, but $N_c = 3$ is clearly ruled out. We can therefore conclude that the decays $\eta, \eta' \to \pi^+\pi^-\gamma$ at leading order are not suited to confirm the number of colors. In the following we investigate whether the situation changes by including next-to-leading order corrections.

At next-to-leading order gauge invariant counter terms of unnatural parity enter the calculation. First, there is a term of fourth chiral order which is suppressed by one order in N_c with respect to the leading order result [8, 10, 11]

$$d^4x\, \tilde{\mathcal{L}}_{p^4} = i\tilde{L}_1 \psi \langle dv\, dU dU^\dagger + dv\, dU^\dagger dU\rangle \tag{14}$$

with $\psi = -i \ln\det U$ and we have neglected for brevity both the external axial-vector fields and the QCD vacuum angle θ.

At the same order in the δ expansion of large N_c ChPT counter terms of sixth chiral order contribute which can be decomposed into explicitly symmetry breaking terms and terms with additional derivatives [11, 12]

$$\tilde{\mathcal{L}}_{p^6} = \tilde{\mathcal{L}}_\chi + \tilde{\mathcal{L}}_{\partial^2}, \tag{15}$$

where

$$\begin{aligned} d^4x\, \tilde{\mathcal{L}}_\chi = {}& \tilde{K}_1\langle \left(U^\dagger\chi - \chi^\dagger U\right)\left[\left(U^\dagger dv U + dv\right) U^\dagger dU U^\dagger dU + U^\dagger dU U^\dagger dU \left(U^\dagger dv U + dv\right)\right]\rangle \\ &+ \tilde{K}_2 \langle \left(U^\dagger\chi - \chi^\dagger U\right) U^\dagger dU \left(U^\dagger dv U + dv\right) U^\dagger dU\rangle \end{aligned} \tag{16}$$

with the mass matrix $\chi = \mathrm{diag}(m_\pi^2, m_\pi^2, 2m_K^2 - m_\pi^2)$ and

$$\begin{aligned} d^4x\, \tilde{\mathcal{L}}_{\partial^2} = {}& \tilde{K}_3\langle (U^\dagger dv U + dv)([U^\dagger \partial^\lambda dU - \left(\partial^\lambda dU\right)^\dagger U] U^\dagger dU\ \ U^\dagger\partial_\lambda U \\ &\qquad + U^\dagger \partial_\lambda U\ \ U^\dagger dU [U^\dagger\partial^\lambda dU - (\partial^\lambda dU)^\dagger U])\rangle \\ &+ \tilde{K}_4\langle (U^\dagger dv U + dv)([U^\dagger \partial^\lambda dU - (\partial^\lambda dU)^\dagger U] U^\dagger dU\ \ U^\dagger\partial_\lambda U \\ &\qquad + U^\dagger \partial_\lambda U\ \ U^\dagger dU [U^\dagger\partial^\lambda dU - (\partial^\lambda dU)^\dagger U])\rangle . \end{aligned} \tag{17}$$

At next-to-leading order we replace the charge matrix Q by $Q^{(0)}$, since $Q^{(1)}$ contributes beyond our working precision. Without mixing the counter terms yield the amplitudes

$$\mathcal{A}^{(ct)}(\phi \to \pi^+\pi^-\gamma) = \frac{8e}{\sqrt{3}f^3} k_\mu \epsilon_\nu p_\alpha^+ p_\beta^- \epsilon^{\mu\nu\alpha\beta}\beta_\phi \tag{18}$$

with the coefficients

$$\begin{aligned} \beta_{\eta_8} &= m_\pi^2\left[2\tilde{K}_1 + \tilde{K}_2\right] - [m_\eta^2 + 2s_{+-} - 2m_\pi^2]\tilde{K}_3 - [s_{+-} - 2m_\pi^2]\tilde{K}_4 \\ \beta_{\eta_0} &= \frac{3}{\sqrt{2}}\tilde{L}_1 + \sqrt{2}m_\pi^2\left[2\tilde{K}_1 + \tilde{K}_2\right] - \sqrt{2}[m_{\eta'}^2 + 2s_{+-} - 2m_\pi^2]\tilde{K}_3 - \sqrt{2}[s_{+-} - 2m_\pi^2]\tilde{K}_4 \end{aligned} \tag{19}$$

and $s_{+-} = (p^+ + p^-)^2$. One must furthermore account for the Z-factors of the mesons and η-η' mixing up to next-to-leading order. For each pion leg the pertinent Z-factor

$$\sqrt{Z}_\pi = 1 - \frac{4}{f^2} m_\pi^2 L_5^{(r)} \tag{20}$$

can be completely absorbed by replacing one factor of f by the physical decay constant F_π in the denominator of the amplitude, Eq. (18),

$$F_\pi = f\left(1 + \frac{4}{f^2} m_\pi^2 L_5^{(r)}\right). \tag{21}$$

The coupling constant $L_5^{(r)}$ originates from the effective Lagrangian of natural parity

$$\mathcal{L}_{eff} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \ldots \tag{22}$$

which reads at lowest order δ^0

$$\mathcal{L}^{(0)} = \frac{f^2}{4}\langle \partial_\mu U^\dagger \partial^\mu U\rangle + \frac{f^2}{4}\langle \chi U^\dagger + U\chi^\dagger\rangle - \frac{1}{2}\tau\psi^2 \tag{23}$$

and at next-to-leading order $\mathcal{O}(\delta)$

$$\begin{aligned}\mathcal{L}^{(1)} &= L_5\langle \partial_\mu U^\dagger \partial^\mu U(\chi^\dagger U + U^\dagger \chi)\rangle + L_8\langle \chi^\dagger U \chi^\dagger U + U^\dagger \chi U^\dagger \chi\rangle \\ &+ \frac{f^2}{12}\Lambda_1 \partial_\mu\psi\partial^\mu\psi + i\frac{f^2}{12}\Lambda_2\psi\langle\chi^\dagger U - U^\dagger\chi\rangle.\end{aligned} \tag{24}$$

Note that both L_5 and L_8 contain divergent pieces which compensate divergencies from loop integrals at order $\mathcal{O}(\delta^2)$ and are thus suppressed by one order in N_c with respect to the finite parts $L_5^{(r)}, L_8^{(r)}$. To the order we are working, we omit the divergent portions.

In the tree level expression for the decay amplitude, Eq. (8), the states η_8 and η_0 are replaced by the physical states η and η' via [7]

$$\begin{aligned}\frac{1}{f}\eta_8 &= \frac{1}{F_\eta^8}[\cos\vartheta^{(1)} - \sin\vartheta^{(0)}\mathcal{A}^{(1)}]\,\eta + \frac{1}{F_\eta^8}[\sin\vartheta^{(1)} + \cos\vartheta^{(0)}\mathcal{A}^{(1)}]\,\eta' \\ \frac{1}{f}\eta_0 &= \frac{1}{F_{\eta'}^0}[\cos\vartheta^{(0)}\mathcal{A}^{(1)} - \sin\vartheta^{(1)}]\,\eta + \frac{1}{F_{\eta'}^0}[\sin\vartheta^{(0)}\mathcal{A}^{(1)} + \cos\vartheta^{(1)}]\,\eta',\end{aligned} \tag{25}$$

where

$$\begin{aligned}\mathcal{A}^{(1)} &= \frac{8\sqrt{2}}{3F_\pi^2}L_5^{(r)}[m_K - m_\pi^2], \\ \sin2\vartheta^{(1)} &= \sin2\vartheta^{(0)}\left(\frac{1+\Lambda_2}{\sqrt{1+\Lambda_1}} + \frac{8}{F_\pi^2}[2L_8^{(r)} - L_5^{(r)}](m_K^2 - m_\pi^2) - \frac{24}{F_\pi^4}L_5^{(r)}\tau\right).\end{aligned} \tag{26}$$

The numerical discussion of these expressions is presented in [7]. For the counter term contributions in Eq. (18), on the other hand, we keep only the leading order pieces in Eq. (25)

$$\begin{aligned}\frac{1}{f}\eta^8 &= \frac{1}{F_\eta^8}\cos\vartheta^{(0)}\eta + \frac{1}{F_\eta^8}\sin\vartheta^{(0)}\eta', \\ \frac{1}{f}\eta^0 &= -\frac{1}{F_{\eta'}^0}\sin\vartheta^{(0)}\eta + \frac{1}{F_{\eta'}^0}\cos\vartheta^{(0)}\eta'\end{aligned} \tag{27}$$

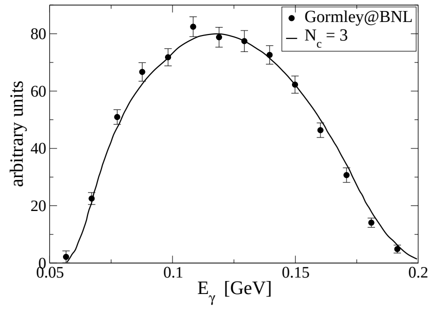

Figure 1: Photon spectrum for $N_C = 3$

which was already employed in the discussion of the leading order decay amplitude, cf. Eq. (12).

From our results it is easy to see that the η' decay does not depend on the QCD renormalization scale. Due to the anomalous dimension of the singlet axial current, the decay constant $F^0_{\eta'}$ scales as, cf. Eq. (9),

$$F^0_{\eta'} \rightarrow Z_A F^0_{\eta'} \ , \tag{28}$$

where Z_A is the multiplicative renormalization constant of the singlet axial current. Furthermore, the $\tilde{K}_i$ are scale independent, whereas $\tilde{L}_1$ transforms according to [8]

$$\tilde{L}_1 \rightarrow \tilde{L}_1^{ren} = Z_A \tilde{L}_1 - \frac{N_C}{144\pi^2}[Z_A - 1]. \tag{29}$$

Since $\tilde{L}_1$ appears in the η' decay amplitude in the combination

$$\left(\frac{N_C}{12\pi^2} - 12\tilde{L}_1\right) \rightarrow \frac{N_C}{12\pi^2} - 12\tilde{L}_1^{ren} = Z_A\left(\frac{N_C}{12\pi^2} - 12\tilde{L}_1\right), \tag{30}$$

the amplitude remains renormalization group invariant.

We now determine the unknown coefficients $\tilde{K}_i$ by fitting them to both the decay width $\Gamma_{\eta\rightarrow\pi^+\pi^-\gamma}$ and the corresponding photon spectrum. To this end, we rewrite the coefficient β_{η_8} in terms of effectively two parameters

$$\beta_{\eta_8} \equiv \beta^{(1)}_{\eta_8} + \beta^{(0)}_{\eta_8} s_{+-}. \tag{31}$$

Setting $N_c = 3$ we obtain a perfect fit to both the experimental decay width $\Gamma_{\eta\rightarrow\pi^+\pi^-\gamma} = 56.1 \pm 5.4$ eV and the photon spectrum, see Fig. 1 , with $\beta^{(1)}_{\eta_8} = 1.3\times10^{-3}$ and $\beta^{(0)}_{\eta_8} = 28.4\times10^{-3}\mathrm{GeV}^{-2}$ which shows that the subleading contributions from the counter terms are important and not suppressed with respect to the leading order originating from the WZW term. However, for $N_c = 2$ an equally good fit to the experimental data, see Fig. 2, is achieved by setting $\beta^{(1)}_{\eta_8} = -3.2\times10^{-3}$ and $\beta^{(0)}_{\eta_8} = 22.0\times10^{-3}\mathrm{GeV}^{-2}$. Although a fit for $N_c = 1$ would be possible as well, we do not present the results here, as a world with $N_c = 1$ has no strong interactions. Note that in the present work we do not explore the possibility of estimating the values of the unknown couplings by means of model-dependent assumptions such as resonance saturation.

It thus does not seem to be possible to strictly determine the number of colors at next-to-leading order in large N_c ChPT, unless one imposes in addition the cancellation of Witten's global $SU(2)_L$ anomaly which requires N_c to be odd [13]. In that case $N_c = 2$ is ruled out

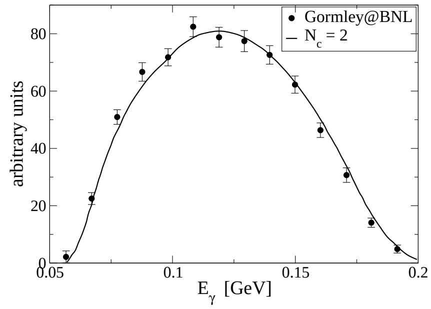

Figure 2: Photon spectrum for $N_C = 2$

and for $N_c = 5$ it turns out that – to the order we are working – one cannot bring the results into agreement with experiment by varying the couplings. In particular, the photon spectrum can only be reproduced with a larger decay width. One may be inclined to argue that the restriction to odd N_c enables a determination of N_c, but it is well-known from the one-loop calculation of this decay in conventional ChPT that the loop contributions reduce the decay width [11, 14]. It is therefore possible that a next-to-next-to-leading order calculation in large N_c ChPT including one-loop corrections can be brought to agreement with experiment also for $N_c = 5$. However, such an investigation is beyond the scope of the present work. In any case, a rigorous statement on the number of colors cannot be made due to the failure of the anomalous contribution from the WZW term to accomodate the decay width for $N_c = 3$ and the presence of unknown couplings.

In the case of the η' decay unitarity effects via final state interactions are dominating [15, 16]. Therefore, a perturbative approach is insufficient to describe the η' decay, and we will refrain from presenting numerical results here.

We conclude that a clean derivation of the number of colors cannot be achieved by investigating the decays $\eta, \eta' \rightarrow \pi^+\pi^-\gamma$. In particular, $\eta \rightarrow \pi^+\pi^-\gamma$ should not be utilized as a textbook example to confirm the number of colors to be $N_c = 3$.

We thank Robin Nißler for useful discussions. Financial support of the DFG is gratefully acknowledged.

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